Fermilab-Pub-82/89-THY
MAD/TH-80

December, 1982

PACS Nos. 12.40.Hh
12.40.Qq
11.15.Tk

THE SVZ METHOD: WHY IT WORKS AND WHY IT FAILS

Bernice Durand and James B. Whitenton*

University of Wisconsin-Madison, Physics Department, Madison, WI 53706

and

L. Durand

Theory Group, Fermi National Accelerator Laboratory, Batavia, IL 60510

and

University of Wisconsin-Madison, Physics Department, Madison, WI 53706†

ABSTRACT

We show in the context of potential theory, first, why the SVZ procedure for calculating hadronic masses succeeds phenomenologically: it provides a semiclassical interpolating formula which correctly relates input masses to output masses; and second, why it fails theoretically: the potential (or QCD) parameters derived from fits to masses are necessarily different from the true parameters.

1. Background

Shifman, Vainshtein, and Zakharov (SVZ) suggested several years ago [1] that one could use the dispersion relations for two-point functions in QCD in combination with first-order perturbation theory and the operator product expansion to relate the masses of quark-antiquark bound states to fundamental parameters in the field theory. The SVZ procedure (to be sketched below) has been used to correlate the masses and spin splittings of a large number of meson [1-4] and baryon [5] states, and to determine the values of such QCD parameters as the gluon condensate parameter $\langle 0|G^a_{\mu\nu}G^{\mu\nu}_a|0\rangle$, thought to play a crucial role in quark confinement. The fits to hadronic masses have been remarkably successful. When the SVZ method is applied, for example, to charmonium, with the gluon condensate parameter adjusted to fit the J/ψ mass, the predictions for the $^1S_0$ and $^3P_{0,1,2}$ masses agree with the observed masses to within 10–30 MeV [3,4].

The striking phenomenological success of the SVZ method has been rather puzzling. The method requires that there be a common region of validity of the operator product expansion (valid for large Q^2 or short Euclidean times) and of a resonance approximation for the dispersion integral (valid for small Q^2 or long Euclidean times). It is not evident that such a region exists. In fact, Bell and Bertlmann [6,7] showed that the SVZ method failed to give accurate ground state energies when applied to realistic potential models for heavy $q\bar{q}$ systems. Alternately, it failed to reproduce the input potential because the potential parameters had to be adjusted to bring the SVZ prediction into agreement with the known ground state energy. In particular, the potential-theory analog of the gluon condensate parameter was underestimated

by a factor $\gtrsim 2$ by this fitting procedure [7]. Essentially the same results were later found by Bradley et al. [8] and Ditsas and Shaw [9] in the large-N_c limit of two-dimensional QCD: the SVZ method again badly underestimated the (known) condensate parameter. However, the method succeeded phenomenologically: Ditsas and Shaw [9] could fit the ground state spectra for different spin-parity states in the two-dimensional "charmonium" and "upsilonium" systems simultaneously using the adjusted parameters.

In this paper, we investigate the theory behind the "exponential moment" version of the SVZ method [1,4,10] using nonrelativistic potential models for the $q\bar{q}$ system. We show why the method succeeds in fitting hadronic energies, and why it fails in predicting energies or determining (QCD) parameters.

2. The SVZ method

The SVZ method is based on the differentiated dispersion relation or spectral representation satisfied by the current-current Green function in QCD,

$$\frac{1}{N!}\left(-\frac{d}{dQ^2}\right)^N \Pi(-Q^2) = \frac{1}{\pi}\int ds\, \frac{\mathrm{Im}\,\Pi(s)}{(s+Q^2)^{N+1}} \quad . \tag{1}$$

Here $\Pi(q^2)$ is the "polarization function" defined schematically by

$$i\int dx\, e^{iq\cdot x} \langle 0|T\,(J_{\cdots}(x),\, J_{\cdots}(0)|0\rangle = (\text{tensor structure})\times \Pi(q^2), \tag{2}$$

where $J_{\cdots}$ is a tensor current with a nonzero projection on the states $|\alpha\rangle$ of interest and

$$\mathrm{Im}\,\Pi(s) = \pi \sum_\alpha |\langle 0|J_{\cdots}(0)|\alpha\rangle|^2\, \delta(s-M_\alpha^2) \quad . \tag{3}$$

By applying a limiting procedure (the "Borel transform") on eq. (1), SVZ obtain the "exponential moment" formula [1,10][1]

$$M(\sigma) = \frac{1}{\pi}\int ds\, e^{-\sigma s}\, \mathrm{Im}\,\Pi(s) \quad . \tag{4}$$

For σ sufficiently large, only the lowest-mass states in eq. (3) (assumed to be discrete) contribute to the integral, and one finds that

$$M(\sigma) \sim \sum Y_n\, e^{-\sigma M_n^2} + \text{continuum contributions.} \tag{5}$$

It is clear, in particular, that the ratio function

$$R(\sigma) = -\frac{d}{d\sigma}\ln M(\sigma) = -M'(\sigma)/M(\sigma) \tag{6}$$

approaches M_1^2 from above for $\sigma \to \infty$.

To make use of eq. (6), SVZ observe that for σ sufficiently small (Q^2 large), one can evaluate $M(\sigma)$ and $R(\sigma)$ directly by using QCD perturbation theory and the operator product expansion in eq. (2), and then applying the Borel transform. The result is an expression [3,4]

$$R_{QCD}(\sigma) = A(\sigma) + \alpha_s B(\sigma) + \alpha_s \langle a\rangle\, C_a(\sigma) + \ldots \tag{7}$$

in which $A(\sigma)$ is the value of R for noninteracting quarks, B is the lowest order perturbative correction, and C_a is the leading nonperturbative correction with condensate parameter $\langle a\rangle$. By assuming, finally, that there is a region of σ in which both the large-σ and small-σ approximations are valid, and requiring that the results be consistent, i.e., that R_{QCD} be approximately constant, $dR_{QCD}/d\sigma = 0$, SVZ obtain a relation for the hadronic mass in terms of the QCD parameters, $M_1^2 = \min_\sigma R_{QCD}(\sigma)$. This relation has been used with known hadronic masses to determine the condensate parameters in eq. (7) (quantities of interest in themselves), and to predict other hadronic masses in terms of those parameters with the striking results noted above.

3. The SVZ method for potential theory

In order to study the SVZ procedure in a well-understood situation, we have applied it to confining potential models for the $q\bar{q}$ system. While our results are therefore most directly relevant for heavy quark systems for which potential models are highly successful, we believe our conclusions have more general validity as suggested, e.g., by the work of Ditsas and Shaw [9].

In a nonrelativistic potential model, the currents $J_{\ell m} \propto Y_{\ell m}(\nabla)$, [2] and the Green function Π is proportional to $\tilde{G}_\ell(0,0,E)$ where [11]

$$\tilde{G}_\ell(r',r,E) = \lim_{r',r\to 0} \frac{1}{(r'r)^\ell} \int_{-\infty}^{\infty} dt\, e^{iEt}\, G_\ell(r',r,t) \tag{8}$$

with

$$G_\ell(r',r,t) = i\theta(t) \sum_{n=1}^{\infty} R_{n\ell}(r')\, e^{-iE_{n\ell}t}\, R^*_{n\ell}(r) \ . \tag{9}$$

Here $R_{n\ell}(r)$ is the radial wave function for the nth state with angular momentum ℓ and $E_{n\ell}$ is the nonrelativistic energy. $\tilde{G}_\ell$ satisfies a dispersion relation which follows from eqs. (8) and (9),

$$\tilde{G}_\ell(0,0,-E) = \frac{1}{\pi} \int_0^{\infty} dE' \, \frac{\mathrm{Im}\tilde{G}_\ell(0,0,E')}{E' + E} \ , \tag{10}$$

with

$$\mathrm{Im}\, \tilde{G}_\ell(0,0,E') = \pi \lim_{r',r\to 0} \frac{1}{(r'r)^\ell} \sum_{n=1}^{\infty} R_{n\ell}(r')R^*_{n\ell}(r)\delta(E-E_{n\ell}) \ . \tag{11}$$

Applying the inverse Laplace transform (or Borel transform[1]) $\mathcal{L}^{-1}(\cdot)(\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dE\, e^{E\tau}(\cdot)$ to eq. (10) and using eq. (11), we obtain the potential-theory analog of eq. (4),

$$M_\ell(\tau) = \frac{1}{\pi}\int_0^{\infty} dE\, e^{-E\tau}\, \mathrm{Im}\, \tilde{G}_\ell(0,0,E)$$

$$= \lim_{r'r\to 0} \frac{1}{(r'r)^\ell} \sum_{n=1}^{\infty} R_{n\ell}(r')\, e^{-E_{n\ell}\tau} R^*_{n\ell}(r) \ . \tag{12}$$

The series in eq. (12) is just an angular projection of the exact Euclidean or imaginary-time propagator for the $q\bar{q}$ system,

$$M_\ell(\tau) = \lim_{r',r\to 0} \frac{1}{(r'r)^\ell} \int d\Omega_{\hat{r}}\, d\Omega_{\hat{r}'} Y_{\ell m}(\hat{r}')\, K(\vec{r}',\vec{r},-i\tau)\, Y_{\ell m}(\hat{r}) \tag{13}$$

where

$$K(\vec{r}',\vec{r},-i\tau) = \langle \vec{r}' | e^{-H\tau} | \vec{r} \rangle \ . \tag{14}$$

We now follow the SVZ argument. It is clear from eq. (12) that the ratio function $R_\ell(\tau) = -\frac{d}{d\tau} \ln M_\ell(\tau)$ approaches $E_{1\ell}$ from above for $\tau \to \infty$. To calculate $M_\ell(\tau)$ directly, we note that the nonrelativistic analog of the Borel-transformed operator product expansion is the <u>short</u> (Euclidean) <u>time</u> perturbation expansion [11,13]. The calculation is easily performed to first order in the (confining) $q\bar{q}$ potential $V(r)$ by iterating the standard integral equation for $K(\vec{r}',\vec{r},-i\tau)$ and using the angular projection in eq. (13). We find that $R_\ell(\tau)$ is given for τ sufficiently <u>small</u> by [11]

$$R^{(1)}_\ell(\tau) = \frac{\ell + \frac{3}{2}}{\tau} + \int_0^{\infty} dz\, H_\ell(z)\, V((\tau z/m_q)^{1/2}) \tag{15}$$

where

$$H_\ell(z) = \frac{\sqrt{\pi}}{\Gamma(\ell+\frac{3}{2})} \sum_{k=0}^{\ell} \frac{\ell!\Gamma(\ell-k-\frac{1}{2})}{k!(\ell-k)!\Gamma(-\frac{1}{2})}\, z^k\, e^{-z} \ . \tag{16}$$

Our best first-order estimate for the energy $E_{1\ell}$ is then obtained by minimizing $R^{(1)}_\ell(\tau)$,

$$E^{(1)}_{1\ell} = \min_\tau [R^{(1)}_\ell(\tau)] \ . \tag{17}$$

As we now show, this procedure has a remarkable hidden structure which makes it reasonably successful despite the inherent inconsistency of using a short-time approximation for R_ℓ to calculate the ground state energy, which is a long-time property of the system.

4. The structure of the SVZ approximation

It is simple to show that the SVZ calculation of the S-state energy E_{1S} gives a variational upper bound on this quantity. The Rayleigh-Ritz variational principle states that

$$E_{1S} \leq \int d^3r\, \phi^*[-\frac{1}{m_q}\nabla^2 + V]\,\phi \qquad (18)$$

for any normalized trial wave function $\phi(r)$. For $\ell=0$, the expression for $R_\ell^{(1)}$ in eq. (15) can be written as

$$R_0^{(1)} = \frac{3}{2\tau} + \frac{m_q^2}{2\pi\tau^2}\int d^3r\; r\; e^{-m_q r^2/\tau}\; V(r). \qquad (19)$$

The integral gives the expectation value of V for the normalized trial wave function

$$\phi(r) = \frac{m_q}{\tau}\left(\frac{\tau}{2\pi}\right)^{1/2} e^{-m_q r^2/2\tau}, \qquad (20)$$

where τ is now identified as the variational parameter. After calculating the expectation value of the kinetic energy for this wave function, and using eq. (18), we find that for any $\tau > 0$ [11,13]

$$E_{1S} \leq R_0^{(1)}(\tau) - \frac{1}{4\tau} < R_0^{(1)}(\tau)\ . \qquad (21)$$

Hence $E_{1S} < \min_\tau [R_0^{(1)}(\tau)] = E_{1S}^{(1)}$, and the SVZ estimate for the energy is always higher than the true energy. (A less general version of this result restricted to a special class of power-law potentials was recently proved by Bertlmann [14].)

This result has two important implications. First, it explains why the SVZ procedure succeeds in giving reasonably accurate S-state energies [6,11] even though the values of τ which minimize $R_o^{(1)}$ are typically large enough that $R_o^{(1)}$ is a poor approximation to R_o: τ is used only as a variational parameter in eq. (17). Its size is irrelevant. The accuracy of the approximation is determined and limited by the (hidden) choice of trial wave function. Second, it explains the empirical observation of Bell and Bertlmann [6,7] that the SVZ energy $E_{1S}^{(1)}$ always lies above the true energy, and shows rigorously that the potential energy must be adjusted downward (e.g., by reducing the analog of the gluon condensate parameter [7]) in any fitting procedure which adjusts the SVZ energy to agree with E_{1S}.

The variational argument does not extend to $\ell > 0$ (the function $H_\ell(z)$ in eq. (15) is only positive definite for $\ell=0$), nor does it hold for higher-order approximations to R_ℓ. However, our numerical calculations [11] show that $H_\ell(z)$ is peaked for $z \sim \ell$, has unit area, and may be approximated for smooth potentials by a delta function at $z \sim \ell + \frac{3}{2}$. With this approximation,

$$R_\ell^{(1)}(\tau) \;\tilde{\approx}\; \frac{\ell+\frac{3}{2}}{\tau} + V\left(\left[(\ell+\tfrac{3}{2})\tau/m_q\right]^{1/2}\right). \qquad (22)$$

A short calculation shows that

$$E_{1\ell}^{(1)} \;\tilde{\approx}\; \min_r\left[V_{eff}(r)\right] \qquad (23)$$

where V_{eff} is the classical effective potential for angular momentum $L = \ell+\frac{3}{2}$,

$$V_{eff}(r) = \frac{L^2}{m_q r^2} + V(r)\ . \qquad (24)$$

This unexpected result can be understood qualitatively on the basis of a path-integral argument [11], and may be regarded as a rough approximation to the JWKB energy for a confining potential. It guarantees that the SVZ energies vary properly with ℓ, the quark mass, and the potential strength, and suggests that if the potential is adjusted to fit the lowest energies, all nearby energies will also be fitted automatically.

More precisely, for confining potentials which can be fitted locally (that is, in the region relevant to some finite set of levels) by a power-law potential $V(r) = V_o + V_1(r/a)^\nu$, $V_1 > 0$, $\nu > 0$, explicit calculation of the integral in eq. (15) and minimization of $R_\ell^{(1)}(\tau)$ gives

$$E_{1\ell}^{(1)} \approx V_o + A(\nu)\left[V_1^{1/\nu}(m_q a^2)^{-1/2}\ (\ell+\tfrac{3}{2})\right]^{2\nu/(2+\nu)} . \qquad (25)$$

This result agrees exactly in the dependence on ℓ, m_q, and V_1 with the n=1 JWKB result for $E_{n\ell}$ derived by Quigg and Rosner [15] for power-law potentials. Only the coefficient $A(\nu)$ is incorrect [11]. The SVZ fitting procedure in effect adjusts V_1 to compensate for the error in $A(\nu)$ by altering the parameters in the given potential. The result is an accurate simultaneous fit to all levels for which the approximate potential is valid, with an inaccurate potential.

5. Success and failure

We summarize our results and conclusions as follows:

(i) The usual first-order SVZ method for calculating the lowest S-state energy E_{1S} is equivalent to a Rayleigh-Ritz variational calculation. As a result, the SVZ prediction for E_{1S} is unexpectedly successful even though the method uses only the leading terms in the large Q^2 operator

product expansion in the small Q^2 region in which the expansion converges very slowly.

(ii) Since the variational prediction for E_{1S} always lies above the true energy, the potential term in the SVZ formula must always be adjusted downward (e.g., by reducing the analog of the gluon condensate parameter) to bring the prediction into agreement with the exact results. The changes necessary may be quite large even when the initial prediction for E_{1S} is fairly accurate [6,7]. The SVZ method therefore fails as a device to obtain reliable potential (or QCD [8,9]) parameters from hadronic masses.

(iii) For smooth potentials, the SVZ predictions for the ground state energies $E_{1\ell}$ for different angular momenta ℓ are connected by an approximate semiclassical relation. This JWKB-like relation guarantees that the predicted energies scale properly with the potential strength and quark mass, and explains the success of the method in fitting hadronic masses. In particular, when the potential (or QCD) parameters are adjusted in the SVZ procedure to fit the lowest energies, e.g., in charmonium, the nearby energies $E_{1\ell}$ and the corresponding energies in upsilonium scale properly and are predicted accurately even though the parameters are incorrect. We illustrate this numerically in Table 1 using the realistic Coulomb-plus-linear potential.

6. Improvements

It is possible to make the SVZ procedure reliable in potential theory by going (at least) to second order in the operator product expansion (this will be discussed elsewhere [11]). This is equivalent in the field-theoretic problem to including (at least) all quadratic terms in the condensates in

the expansion, a formidable task, and is complicated by the appearance of new condensates when high-dimension terms are included.

FOOTNOTES

*Present address: Physics Department, Southern Technical Institute, Marietta, Georgia 30060

†Permanent address

1. Specifically, $M(\sigma) = \lim_{\substack{Q^2,N\to\infty \\ N/Q^2=\sigma}} \frac{(Q^2)^{N+1}}{N!}\left(-\frac{d}{dQ^2}\right)^N \Pi(-Q^2)$.

   The result is completely equivalent to that obtained by applying an inverse Laplace transform to Π, ignoring the additive constant which appears in eq. (1) for $N=0$,

$$M(\sigma) = L^{-1}(\Pi)(\sigma) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dQ^2\, e^{\sigma Q^2}\, \Pi(-Q^2) .$$

2. We neglect spin and consider only the orbital excitations of the $q\bar{q}$ system.

REFERENCES

1. M.A. Shifman, A.I. Vaĭnshteĭn, and V.I. Zakharov, Nucl. Phys. B147 (1979) 385, 448, 519; Phys. Rev. Lett. 42 (1979) 297.

2. M.A. Shifman, A.I. Vaĭnshteĭn, M. Voloshin, and V.I. Zakharov, Phys. Lett. 77B (1978) 80.

3. L.J. Reinders, H.R. Rubenstein, and S. Yazaki, Phys. Lett. 94B (1980) 203; 95B (1980) 103; 103B (1981) 63; 104B (1981) 305; Nucl. Phys. B186 (1981) 109.
   L.J. Reinders, S. Yazaki, and H.R. Rubenstein, Nucl. Phys. B196 (1982) 125.

4. R.A. Bertlmann, Phys. Lett. 106B (1981) 336; Nucl. Phys. B204 (1982) 387.
   B. Guberina, R. Mechbach, R.D. Peccei, and R. Rückl, Nucl. Phys. B184 (1981) 476.

5. B.L. Ioffe, Nucl. Phys. B188 (1981) 317.
   Y. Chung, H.G. Dosch, M. Kremer, and D. Schall, Phys. Lett. 102B (1981) 175.

6. J.S. Bell and R.A. Bertlmann, Nucl. Phys. B177 (1981) 218.
   R.A. Bertlmann, Acta Phys. Aust. 53 (1981) 305.

7. J.S. Bell and R.A. Bertlmann, Nucl. Phys. B187 (1981) 285.

8. A. Bradley, C.S. Langensiepen, and G. Shaw, Phys. Lett. 102B (1981) 359.

9. P. Ditsas and G. Shaw, Phys. Lett. 116B (1982) 188.

10. A.I. Vaĭnshteĭn, V.I. Zakharov, V. A. Novikov, and M.A. Shifman, Soviet J. Nucl. Phys. 32 (1980) 840 (translation of Yad. Fiz. 32 (1980) 1622).
    V.A. Novikov, M.A. Shifman, A.I. Vaĭnshteĭn, and V.I.Zakharov, Nucl. Phys. B191 (1981) 301.

11. L. Durand, B. Durand, and J.B. Whitenton, University of Wisconsin/ Fermilab preprint MAD/TH-76/Fermilab-Pub-82/88-THY, 1982.

12. J.B. Whitenton, B. Durand and L. Durand, University of Wisconsin/ Fermilab preprint MAD/TH-42/Fermilab-Pub-82/90-THY, 1982.

13. J.B. Whitenton, University of Wisconsin-Madison dissertation, 1982 (unpublished).

14. R.A. Bertlmann, CERN preprint TH.3440, October, 1982.

15. C. Quigg and J. Rosner, Phys. Rep. 56 (1979) 167.

Table 1. Comparison of the exact ground state energies for the Coulomb-plus linear potential $V = -ar^{-1}+br$ with the parameters of K.J. Miller and M.G. Olsson, Phys. Rev. D 25 (1982) 2383, with the first order energies calculated by minimizing the SVZ exponential moments function $R_\ell^{(1)}$. The masses of the charm and bottom quark are m_c=1.35 GeV, m_b=4.77 GeV. The last two columns give the results obtained when the potential parameters are modified so that the first order 1S and 1P charmonium energies are correct.

State	a=0.49, b=0.17 GeV^2			a=0.665, b=0.146 GeV^2	
	$E_{1\ell}$,exact (MeV)	$E_{1\ell}^{(1)}$ (MeV)	Error (MeV)	$E_{1\ell}^{(1)}$ (MeV)	Error (MeV)
Charmonium					
1S	364	508	+144	364	fitted
1P	772	914	+142	772	fitted
1D	1060	1221	+161	1063	+3
1F	1305	1487	+182	1314	+9
Upsilonium					
1S	-98	106	+204	-94	+4
1P	349	479	+130	343	-6
1D	585	714	+129	581	-4
1F	769	905	+136	767	-2